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THE FILTERING OF SOUND WAVES FROM THE NON-HYDROSTATIC EQUATIONS FOR ATMOSPHERIC MOTION

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1. INTRODUCTION

The full non-hydrostatic equations of motion support both sound waves and gravity waves. Vertically propogating sound waves can be filtered from these equations by making the hydrostatic approximation, which is satisfied to a high degree of accuracy over nearly all scales. However, the dispersion relation for gravity waves, and thus their propagation properties, are seriously altered by making the hydrostatic assumption even for quite low aspect ratios (we refer to the ratio of the vertical and horizontal wavelengths as the aspect ratio) and the general applicability of a model that is intended for use on sub synoptic scales is restricted if it is used.

The other commonly considered approximation that removes sound waves is the anelastic approximation. As the name implies, this eliminates sound waves by excluding elastic compression of the air, and this approximation is very satisfactory for cumulus studies, for which it was designed, and also for studies of very shallow systems. However, the degree of approximation involved is greater than for the hydrostatic assumption, and calculation using the anelastic equations must be expected to be numerically inaccurate except in a narrow range of applications.

In this Note I propose an approximation that has not, as far as I know, been considered before. It is very similar to the anelastic approximation in the effect it has on the properties of the full equations, but it does not involve the wholesale neglect of perturbation terms and is thus accurate for a far wider range of applications. The new approximation should be good for all tropospheric systems with horizontal scales less than 10^3 km, and it is not proved that it is inappropriate on larger scales.

In Section 2 the basic atmospheric equations are introduced in a convenient form, and the dispersion relation for sound and gravity waves is introduced in order to show that a clear distinction can be drawn between these two systems. An interesting feature of the present approximation is that it eliminates Lamb waves,

sometimes called the external gravity wave but in fact a compression wave, and it is an open question (for me at least) whether or not this limits its applicability.

The anelastic approximation is presented and discussed in Section 3. I hope that this will prove a useful reference for those who are curious about the anelastic equations and, like me, find the original paper by Ogura and Phillips (J.A.S. 19, p 173, 1962) difficult to obtain.

Section 4 contains a brief discussion, and Section 5 contains an exposition of the new approximation.

2. SOUND AND GRAVITY WAVES

The equations of motion, expressed in terms of the Exner function $P \equiv (p/p_r)^K$ and the potential temperature $\theta = T/P$ and ignoring friction, are

$$\frac{Du}{Dt} - fv = -c_p \theta \frac{\partial P}{\partial x} \qquad 1$$

$$\frac{Dv}{Dt} + fu = -c_p \theta \frac{\partial P}{\partial y} \qquad 2$$

$$\frac{Dw}{Dt} + g = -c_p \theta \frac{\partial P}{\partial z} \qquad 3$$

$$\frac{D\theta}{Dt} = Q/c_p P \qquad 4$$

$$\frac{1}{(\gamma-1)} \frac{1}{P} \frac{DP}{Dt} - \frac{1}{\theta} \frac{D\theta}{Dt} = -\nabla \cdot \underset{\sim}{u} \qquad 5$$

Write

$$\theta = \theta_0 + \theta_1 \qquad 6$$

where we allow

$$\theta_0 = \theta_0(z) \qquad 7$$

and introduce

$$\frac{\partial P_0}{\partial z} = -\frac{g}{c_p \theta_0} \qquad 8$$

with

$$P_0 = 1 \quad \text{for} \quad z = 0$$

defining P_1 by

$$P = P_0 + P_1 \qquad 9$$

Equations 1 to 5 can now be written

$$\frac{Du}{Dt} - fv = -c_p\theta \frac{\partial p_1}{\partial x} \quad 10$$

$$\frac{Dv}{Dt} + fu = -c_p\theta \frac{\partial p_1}{\partial y} \quad 11$$

$$\frac{Dw}{Dt} - g\frac{\theta_1}{\theta_0} = -c_p\theta \frac{\partial p_1}{\partial z} \quad 12$$

$$w\frac{\partial \theta_0}{\partial z} + \frac{D\theta_1}{Dt} = \frac{Q}{c_p T} \quad 13$$

$$\frac{1}{(\gamma-1)T}\frac{Dp_1}{Dt} - \frac{1}{\theta}\frac{D\theta}{Dt} = \frac{gw}{(\gamma-1)c_p\theta_0 p_0} - \nabla\cdot\underset{\sim}{u} \quad 14$$

Equations 10 to 14 are those used in the Met O 11 mesoscale model, except that θ_0 is independent of z changing equation 13 to

$$\frac{D\theta_1}{Dt} = \frac{Q}{c_p T} \quad 15$$

These equations support sound waves and gravity waves. If we neglect the non-linear inertial terms and the variation of density, but not buoyancy, with height we obtain the linearised equations

$$\frac{\partial u}{\partial t} - fv = -c_p\bar{\theta}\frac{\partial p'}{\partial x} \quad 16$$

$$\frac{\partial v}{\partial t} + fu = -c_p\bar{\theta}\frac{\partial p'}{\partial y} \quad 17$$

$$\frac{\partial w}{\partial t} - g\frac{\theta'}{\bar{\theta}} = -c_p\bar{\theta}\frac{\partial p'}{\partial z} \quad 18$$

$$\frac{\partial \theta'}{\partial t} + w\frac{\partial \bar{\theta}}{\partial z} = 0 \quad 19$$

$$\frac{c_p\bar{\theta}}{c^2}\frac{\partial p_1}{\partial t} = \frac{g}{c^2}w - \nabla\cdot\underset{\sim}{u} \quad 20$$

where diabatic and frictional effects have been excluded and the $\bar{}$ and prime notation for the linearisation about a basic state and have been introduced in order to avoid confusion with the suffix 0 and suffix 1 notation, which describes a different manipulation.

$$c^2 = (\gamma-1)c_p\bar{p}\,\bar{\theta} \quad \text{approximated constant} \quad 21$$

Following the usual procedure of searching for a solution $\propto f(z)e^{i(\sigma t - kx - ly)}$ we find $f(z) \propto e^{-\mu z + imz}$

where

$$2\mu = \frac{g}{(\gamma-1)c_p\bar{p}\,\bar{\theta}} = \frac{g}{c^2} \quad 22$$

and obtain the dispersion relation

$$\sigma^4 - \sigma^2 [f^2 + N^2 + c^2 (k'^2 + l'^2 + m'^2 + \mu^2)] + [N^2 c^2 (k'^2 + l'^2) + f^2 c^2 (m'^2 + \mu^2)] = 0 \qquad 23$$

where

$$N^2 = \frac{g}{\theta} \frac{\partial \theta}{\partial z} \qquad 24$$

There are always two solutions, σ_g^2 and σ_s^2

$$\sigma_g^2 \approx N^2 \frac{k'^2 + l'^2}{k'^2 + l'^2 + m'^2 + \mu^2} + f^2 \frac{m'^2 + \mu^2}{k'^2 + l'^2 + m'^2 + \mu^2} = \Sigma_g^2 \qquad 25$$

$$\sigma_s^2 \approx c^2 (k'^2 + l'^2 + m'^2 + \mu^2) + N^2 \frac{m'^2 + \mu^2}{k'^2 + l'^2 + m'^2 + \mu^2} + f^2 \frac{k'^2 + l'^2}{k'^2 + l'^2 + m'^2 + \mu^2} = \Sigma_s^2 \qquad 26$$

where the approximations 25 and 26 are obtained by assuming

$$f^2 + N^2 << c^2 (k'^2 + l'^2 + m'^2 + \mu^2) \qquad 27$$

[Typically, $\frac{N^2}{c^2 \mu^2} = 0.4$ in the atmosphere]

In any event

$$(\sigma^2 - \Sigma_s^2)(\Sigma_g^2 - \sigma^2) = - \left(N^2 \frac{m'^2 + \mu^2}{k'^2 + l'^2 + m'^2 + \mu^2} + f^2 \frac{k'^2 + l'^2}{k'^2 + l'^2 + m'^2 + \mu^2} \right) \Sigma_g^2 \qquad 28$$

so that there is always one solution

$$\sigma^2 > \Sigma_s^2 > c^2 (k'^2 + l'^2 + m'^2 + \mu^2)$$

corresponding to sound waves, and a second solution

$$\sigma^2 < \Sigma_g^2 < N^2$$

(assuming $N^2 > f^2$) corresponding to gravity waves.

3. ANELASTIC APPROX

Ogura and Phillips (1962) made a scale analysis of equations 10, 11, 12, 15 and 14 (constant θ_o), and introduced an approximation that Charney called "anelastic". They were considering convective problems and, without explicitly stating it, assumed that the horizontal and vertical length scales were comparable. Defining

$$\underset{\sim}{x}' d = \underset{\sim}{x}$$
$$\underset{\sim}{u}' d/\tau = \underset{\sim}{u}$$
$$\theta_o \phi = \theta_1$$
$$t' \tau = t$$

they obtained

$$\frac{d^2}{\tau^2 c_p \theta_0} \frac{Du'}{Dt'} = -(1+\phi) \frac{\partial P_1}{\partial x} \qquad 29$$

$$\frac{d^2}{\tau^2 c_p \theta_0} \frac{Dv'}{Dt'} = -(1+\phi) \frac{\partial P_1}{\partial y} \qquad 30$$

$$\frac{d^2}{\tau^2 c_p \theta_0} \frac{Dw'}{Dt'} - \frac{g d}{c_p \theta_0} \phi = -(1+\phi) \frac{\partial P_1}{\partial z} \qquad 31$$

$$\frac{D\phi}{Dt} = 0 \qquad 32$$

$$\frac{1}{(\gamma-1) P} \frac{DP_1}{Dt} = \frac{g d}{(\gamma-1) c_p \theta_0} w' - \underset{\sim}{\nabla}' \cdot \underset{\sim}{u}' \qquad 33$$

where coriolis and diabatic effects were neglected. Two assumptions were then stated:

1) ϕ is small
2) $\tau \sim N^{-1}$

where $$N^2 = \frac{g}{\theta} \frac{\partial \theta}{\partial z} \sim \frac{g \phi}{d}$$

The second assumption gives

$$\phi \sim \frac{d}{g \tau^2} \qquad 34$$

and states that gravity waves are of prime interest, but sound waves should be filtered from the equations.

Writing

$$\varepsilon = \frac{d}{g \tau^2} \qquad 35$$

$$\beta = \frac{g d}{c_p \theta_0} \qquad 36$$

gave

$$\varepsilon \beta \frac{Du}{Dt} = -(1+\phi) \frac{\partial P_1}{\partial x} \qquad 37$$

$$\varepsilon \beta \frac{Dv}{Dt} = -(1+\phi) \frac{\partial P_1}{\partial y} \qquad 38$$

$$\varepsilon \beta \frac{Dw}{Dt} = \beta \phi - (1+\phi) \frac{\partial P_1}{\partial z} \qquad 39$$

$$\frac{1}{(\gamma-1) P} \frac{DP_1}{Dt} = \frac{\beta}{(\gamma-1)} w - \underset{\sim}{\nabla} \cdot \underset{\sim}{u} \qquad 40$$

where primes are now dropped. From 34, 35, 37, 38 and 39,

$$\nabla P_1 \sim \varepsilon$$

Equation 40 shows that, in the absence of net inflow into a fixed volume, the volume average of $\frac{\partial P_1}{\partial t}$ vanishes, so Ogura and Pillips inferred

$$P_1 \sim \varepsilon \qquad 41$$

The anelastic approximation is now made by neglecting, in each equation, all terms that are not of the lowest possible order in ε. This gives

$$\varepsilon \beta \frac{Du}{Dt} = -\frac{\partial P_1}{\partial x} \qquad 42$$

$$\varepsilon \beta \frac{Dv}{Dt} = -\frac{\partial P_1}{\partial y} \qquad 43$$

$$\varepsilon \beta \frac{Dw}{Dt} = \beta\phi - \frac{\partial P_1}{\partial z} \qquad 44$$

$$\frac{\beta w}{(\gamma-1)} = \nabla \cdot \underset{\sim}{u} \qquad 45$$

$$\frac{D\phi}{Dt} = 0 \qquad 46$$

or, returning to fully dimensional notation,

$$\frac{Du}{Dt} - fv = -c_p \theta_0 \frac{\partial P_1}{\partial x} \qquad 47$$

$$\frac{Dv}{Dt} + fu = -c_p \theta_0 \frac{\partial P_1}{\partial y} \qquad 48$$

$$\frac{Dw}{Dt} = g\frac{\theta_1}{\theta_0} - c_p \theta_0 \frac{\partial P_1}{\partial z} \qquad 49$$

$$\frac{D\theta_1}{Dt} = \frac{Q}{c_p P_0} \qquad 50$$

$$\frac{gw}{(\gamma-1)c_p \theta_0 P_0} = \nabla \cdot \underset{\sim}{u} \qquad 51$$

where coriolis and diabatic terms have been restored (by me, not Ogura and Phillips).

Equations 47 to 51 support gravity waves, but not sound waves. Deriving a **dispersion** relation in exactly the same way as in the previous section we obtain

$$(m^2+\mu^2)(\sigma^2-f^2) = (k^2+l^2)(N^2-\sigma^2)$$

$$\text{i.e.} \quad \sigma^2 = N^2 \frac{k^2+l^2}{k^2+l^2+m^2+\mu^2} + f^2 \frac{m^2+\mu^2}{k^2+l^2+m^2+\mu^2} \qquad 52$$

which is the same as the approximate equation 25. If we return to equation 25, we can improve the approximation thus finding the error involved in making the anelastic approx.

$$\sigma_s^2 \simeq c^2(k'^2+l'^2+m'^2+\mu^2) + N^2\frac{m'^2+\mu^2}{k'^2+l'^2+m'^2+\mu^2} + f^2\frac{k'^2+l'^2}{k'^2+l'^2+m'^2+\mu^2}$$

$$\simeq \Sigma_s^2\left(1 - \frac{N^2(m'^2+\mu^2) + f^2(k'^2+l'^2)}{c^2(k'^2+l'^2+m'^2+\mu^2)^2}\right)$$

The error is clearly small. In fact, the largest proportional error is given by $m^2 = 0$ and is less than 0.1 for length scales less than 100 km and always less than 0.1 for typical tropospheric values of m^2 ie $m'^2+\mu^2$, or larger.

Momentum and energy are both fully conserved by equations 47 to 51, providing suitable redefinitions are made. If

$$\rho \simeq \frac{p_r}{R}\, p_0^{\frac{1}{\gamma-1}}\, \theta_0^{-1} \qquad 53$$

$$p = p_r\left(p_0^{\frac{\gamma}{\gamma-1}} + \frac{\gamma}{\gamma-1}\, p_0^{\frac{1}{\gamma-1}}\, p_1\right) \qquad 54$$

equation 51 can be written

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot \rho \underset{\sim}{u} = 0 \qquad 55$$

[noting that $\frac{\partial \rho}{\partial t} = 0$], which is the usual form of the continuity equation. Multiplying equations 47, 48 and 49 by ρ gives the equation for conservation of momentum $\rho \underset{\sim}{u}$

$$\frac{\partial \rho u}{\partial t} + \underset{\sim}{\nabla}\cdot \underset{\sim}{u}\rho u - \rho f v = -c_p \frac{p_r}{R}\, p_0^{\frac{1}{\gamma-1}}\frac{\partial p_1}{\partial x} = -\frac{\partial p}{\partial x} \qquad 56$$

$$\frac{\partial \rho w}{\partial t} + \underset{\sim}{\nabla}\cdot \underset{\sim}{u}\rho w = \rho g\,\theta_1/\theta_0 - \frac{\gamma}{\gamma-1} p_r\, p_0^{\frac{1}{\gamma-1}}\frac{\partial p_1}{\partial z}$$

$$= -\rho g\left(1 + \frac{1}{\gamma-1}\frac{p_1}{p_0} - \frac{\theta_1}{\theta_0}\right) - \frac{\partial p}{\partial z} \qquad 57$$

The extra phrase multiplying the gravitational acceleration term in equation 57 is a first order correction for the difference between the true density $\frac{p_r}{R} p^{\frac{1}{\gamma-1}} \theta^{-1}$ and ρ defined by equation 53. Conservation of energy (or, more correctly, enthalpy) can also be proved in the form

$$\frac{\partial \rho E}{\partial t} + \underset{\sim}{\nabla}\cdot(\underset{\sim}{u}\rho E) = \frac{\partial p}{\partial t} + \rho Q \qquad 58$$

where $$E = \frac{u'^2+v'^2+w'^2}{2} + c_p(p_0\theta_1 + \theta_0 p_1 + \theta_0 p_0) + gz \qquad 59$$

[The simplest derivation of equation 58 proceeds from the l.h.s. to the r.h.s., using the continuity equation to remove ρ from the derivatives and then using the equation of motion to evaluate each term.] Note that the definition of temperature must be modified

$$T = p_0\theta_0 + p_0\theta_1 + \theta_0 p_1 \qquad 60$$

and that the conservation of energy gives equation 50 rather than

$$\frac{1}{\theta}\frac{D\theta_1}{Dt} = \frac{Q}{c_p T} \qquad 61$$

or some other complicated alternative.

4. DISCUSSION

The anelastic equations 47 to 51 have the advantage of being simple and thus amenable to theoretical analysis and rapid computation while including nearly all the important dynamical aspects of the exact equations 1 to 5. There is a nasty folk lore to the effect that Lamb waves (often called the external gravity wave if pressure is being used as the vertical coordinate) are meteorologically important, and, on the largest scales, they cannot be excluded by a scale analysis. Apart from the Lamb wave, I do not know of any physical process described by equations 1 to 5 that is not also, at least qualitatively, described by equations 47 to 51. Thus the anelastic equations can and are used in a far wider range of applications than intended by Ogura and Phillips or, as far as I know, can be justified strictly by a scale analysis.

The anelastic equations are an attractive possibility for the mesoscale modelling group because, in spite of the implicit treatment of sound waves, it is the sound waves that lead to the limit

$$N\delta t \leq 1$$

on the time step, and because no structural alterations (but massive deletions) are required to the mesoscale model code in order to use them. Nevertheless, the first assumption [(1), ϕ is small] of the anelastic equations is only valid for shallow systems or near neutral lapse rates, and the terms that have been neglected in the anelastic equations might be quantitatively important in many situations. While it would be useful and interesting to experiment with the use of the anelastic equations, there is a greater chance of producing good forecasts from some other approximation that eliminates sound waves in the same way as the anelastic equations, but avoids most of its quantitative errors.

5. GENERALISED ANELASTIC APPROX

The anelastic equations do not support sound waves because the possibility of adiabatic compression is eliminated. Therefore, I propose the following set of equations, which are identical to equations 10 to 14 except for the elimination of the adiabatic compression term in the continuity equation.

$$\frac{Du}{Dt} - fv = -c_p\theta \frac{\partial \pi_1}{\partial x} \qquad 10$$

$$\frac{Dv}{Dt} + fu = -c_p\theta \frac{\partial \pi_1}{\partial y} \qquad 11$$

$$\frac{Dw}{Dt} - g\frac{\theta_1}{\theta_0} = -c_p\theta \frac{\partial \pi_1}{\partial z} \qquad 12$$

$$w\frac{\partial \theta_0}{\partial z} + \frac{D\theta_1}{Dt} = \frac{Q}{c_p\pi} \qquad 13$$

$$-\frac{1}{\theta}\frac{D\theta}{Dt} = \frac{g w}{(\gamma-1)c_p\theta_0\pi_0} - \underset{\sim}{\nabla}\cdot\underset{\sim}{u} \qquad 61$$

The only property of this set of equations that must be verified is that they conserve energy and momentum. To this end, define

$$\rho = \frac{p_r}{R}\pi_0^{\frac{1}{\gamma-1}}\theta^{-1} \qquad 62$$

so that equation 61 becomes

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot\rho\underset{\sim}{u} = 0 \qquad 63$$

multiplying 10 by ρ gives

$$\frac{\partial \rho u}{\partial t} + \underset{\sim}{\nabla}\cdot\underset{\sim}{u}\rho u - f\rho v = -\frac{\partial p}{\partial x} \qquad 64$$

where

$$p = p_r\left(\pi_0^{\frac{\gamma}{\gamma-1}} + \frac{\gamma}{\gamma-1}\pi_0^{\frac{1}{\gamma-1}}\pi_1\right) \qquad 54$$

(as in the anelastic approx), and similarly for equation 11, and equation 12 can be manipulated into the form

$$\frac{\partial \rho w}{\partial t} + \underset{\sim}{\nabla}\cdot\underset{\sim}{u}\rho w = g\rho\,\theta_1/\theta_0 - \frac{\gamma}{\gamma-1}p_r\pi_0^{\frac{1}{\gamma-1}}\frac{\partial \pi_1}{\partial z}$$

$$= -\frac{\partial p}{\partial z} - g\rho\left(1 + \frac{1}{\gamma-1}\frac{\pi_1}{\pi_0}\frac{\theta}{\theta_0}\right) \qquad 65$$

Finally, we derive the equation for the conservation of enthalpy, doing so in some detail on this occasion.

$$\frac{D}{Dt}\frac{u'+v'+w'}{2} = g\frac{\theta_1}{\theta_0}w - c_p\theta\,\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\pi_1 \qquad 66$$

by multiplying equations 10, 11 and 12 by u, v and w. If

$$E = \frac{u'+v'+w'}{2} + c_p\pi\theta + gz \qquad 67$$

$$\frac{DE}{Dt} = g\frac{\theta_1}{\theta_0} w - c_p\theta\, \underset{\sim}{u}\cdot\underset{\sim}{\nabla} P_1 + c_p\theta \frac{DP}{Dt} + c_p P \frac{D\theta}{Dt} + gw$$

$$= g\frac{\theta_1}{\theta_0} w - c_p\theta\, \underset{\sim}{u}\cdot\underset{\sim}{\nabla} P_1 + c_p\theta \frac{\partial P_1}{\partial t} + c_p\theta\, \underset{\sim}{u}\cdot\underset{\sim}{\nabla} P_1 + c_p\theta w \frac{\partial P_0}{\partial t} + Q + gw$$

$$= g\frac{\theta_1}{\theta_0} w + c_p\theta \frac{\partial P_1}{\partial t} + Q - c_p\theta \frac{g}{c_p\theta_0} w = c_p\theta \frac{\partial P_1}{\partial t} + Q \qquad 68$$

Immediately

$$\frac{\partial}{\partial t} \rho E + \underset{\sim}{\nabla}\cdot \underset{\sim}{u}\, \rho E = \rho \frac{DE}{Dt} \qquad \text{(using 63)}$$

$$= c_p \rho\theta \frac{\partial P_1}{\partial t} + \rho Q$$

$$= \frac{\partial P}{\partial t} + \rho Q \qquad 67$$

Thus equations 10 to 61 are energetically consistent and are approximate only because the term $\frac{1}{(\gamma-1)}\frac{1}{P}\frac{DP_1}{Dt}$ is excluded, (and, not mentioned above, because the map factor and variation of the coriolis parameter with latitude are ignored). The dynamical effect of the continuity equation is, in most circumstances, to determine the vertical velocity w. The scales over which the present approximation is valid will be those where

$$\frac{1}{(\gamma-1)P}\frac{DP_1}{Dt} \Big/ \frac{\partial w}{\partial t} \ll 1 \qquad 70$$

Introducing the same scaling as before and writing ΔP_1 for typical variations in P_1, equation 70 can be written

$$\frac{\Delta P_1}{\tau} \Big/ \frac{1}{\tau} \ll (\gamma-1)P$$

But $P \approx 1$ at the surface, and $P > \frac{1}{2}$ for heights less than about 15 km, so

$$\Delta P_1 \ll 1/10 \qquad 71$$

is a more stringent inequality than 70. If 71 is satisfied for large enough horizontal scales, we can use the geostrophic relation

$$|\nabla P_1| \simeq f|u|/c_p\theta$$

to determine the part of ΔP_1 due to horizontal variation. In this way, equation 71 levels to an upper limit ℓ on the horizontal scales over which 70 applies

$$\Delta P_1 = \ell\,|\nabla P_1| \approx \frac{\ell f |u|}{c_p\theta}$$

$$l \ll \frac{1}{10}\,\frac{c_p\theta}{g|u|} \sim \frac{10^3\,300}{10\times10^{-4}\times30} \text{ m} = 10000 \text{ km}$$

The part of ΔP_1 due to vertical variations can be estimated using the hydrostatic relation

$$\frac{\partial P_1}{\partial z} = \frac{1}{c_p\theta}\, g\, \frac{\theta_1}{\theta_0} \qquad 72$$

and an estimate of θ_1

$$\theta_1 \sim l\,|\nabla_H \theta| + d\,\left|\delta \frac{\partial\theta}{\partial t}\right| \qquad 73$$

where $\delta\left|\frac{\partial\theta}{\partial t}\right|$ is the difference between the local lapse rate and the basic lapse rate $\frac{\partial\theta_0}{\partial t}$. From equations 72 and 73

$$\Delta P_1 \sim \frac{d}{c_p\theta}\left(\frac{g}{\theta}\, l\, \nabla_H\theta + \frac{g}{\theta}\, d\, \delta\left|\frac{\partial\theta}{\partial t}\right|\right)$$

and so we require

$$\frac{dl}{c_p\theta}\,\frac{g}{\theta}\,\nabla_H\theta \ll \frac{1}{10} \qquad 74$$

and

$$\frac{d^2}{c_p\theta}\,\delta N^2 \ll \frac{1}{10} \qquad 75$$

75 gives

$$d^2 \ll 10^3\, 300\, \frac{1}{10}\, 10^4 \qquad 77$$

ie

$$d \ll 17 \text{ km} \qquad 76$$

where we have over estimated δN^2 by ignoring the basic lapse rate $\frac{\partial\theta_0}{\partial t}$.

Taking d ~ 10 km, equation 74 gives

$$l \ll c_p\theta\,\frac{1}{10}\,\frac{1}{10^4}\,\frac{\theta}{g}\,\frac{1}{\nabla_H\theta} \simeq \frac{1}{10\,\nabla_H\theta} \text{ km}$$

A variation of 10K in 100 km would represent a sharp frontal zone, and this high estimate for $\nabla_H\theta$ gives

$$l \ll \frac{1}{10\frac{10}{10^5}} = 1000 \text{ km}$$

6. CONCLUSION

The scale analysis in the previous section suggests that the new approximation introduced in this Note (equation 61) will probably break down for horizontal scales greater than 1000 km. The limit on the vertical scale expressed in equation 76 can be extended by including the mean vertical variation of potential temperature in the basic potential temperature function $\theta_0(z)$, and, in any case, it accommodates the lower troposphere without modification. The suggestion that the approximation is

not valid for synoptic scale motions is in accordance with the folk lore that the Lamb wave is important on such scales.

By removing the adiabatic compression term, we eliminate all sound waves from the equations and this could allow substantially increased time steps to be used in the mesoscale model. With no change in the finite difference scheme used, the stability criterion would become

$$\frac{N^2 \delta t^2}{1 + \frac{\pi^2}{8}\frac{\delta x^2}{H^2} + \frac{1}{32}\frac{g^2 \delta x^2}{c^4}} \leq 1$$

allowing $\delta t \simeq 5$ mins in the present sea breeze studies, where δx = 10 km and H = 4 km. With δx = 20 km and H=20 km (to allow the upward propogation of wave energy, not because 20 km is the vertical scale of interest) it might be possible to use δt = $2\frac{1}{2}$ mins. It should also be possible to eliminate the time averaging involved in the present implicit treatment of sound waves.

The new approximation is a form of anelastic approximation, but it has a far wider range of validity than was ever intended by Ogura and Phillips for their approximation. It also has the same linearised properties as the anelastic approximation. The distinction between the two sets of equations is that the anelastic equations neglect a large number of product terms and thus facilitate rapid calculation and easy interpretation, while the present approximation retains these terms and is thus, in principle, capable of accurate forecasting.